Polygonal finite elements of arbitrary order

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***Abstract*—Finite elements with the shape of arbitrary polygons have been previously described. Originally they were first order, i.e., able to represent exactly all polynomials of degree 1 in the space coordinates, but recently polygonal finite elements (PFEs) up to order 3 have been reported. Here we propose a general theory for generating PFEs of arbitrary order. Results for a wave problem and a magnetic field problem show the effectiveness of the elements up to order 5.**

***Index Terms*—Computational electromagnetics, finite element analysis, magnetostatics.**

I. Introduction

Despite the success of the triangle and quadrilateral in 2D finite element analysis (FEA), researchers have considered other shapes, including arbitrary polygons [1][2]. Polygonal finite elements (PFEs) can reduce the number of artificial element boundaries in a problem and improve the smoothness of the solution. They also allow movement of parts without introducing discontinuities caused by remeshing.

Given the geometric flexibility of PFEs, it is often possible to model a geometry with a small number of them. However, the solution in that case will only be accurate if the element is high order, i.e., able to represent exactly potentials that are high order polynomials in x and y. Second and third order PFEs were introduced in [3] and [4], respectively. In the next section we give a procedure for generating PFEs of arbitrary order, p. Only an outline is given; details will follow in the full paper.

II. Outline of the Theory

We assume that the PFE has $N \geq \max(4, p+1)$ nodes, with coordinates (x_i, y_i), numbered in sequence around the boundary. For $p = 1$, the basis functions are a set of barycentric coordinates λ_i, one per node, with the properties: they sum to 1 at every point; they can exactly represent any polynomial of degree 1 in x and y; λ_i is 1 at node i and 0 at all other nodes; λ_i is degree 1 in x and y along each edge. Various barycentric coordinates are possible; we use metric coordinates [2]. We define 3 polynomials in the λ_is:

$$Q_1(\lambda) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \lambda_i\,;\;\; Q_2(\lambda) \stackrel{\text{def}}{=} \sum_{i=1}^{N} x_i\lambda_i\,;\; Q_3(\lambda) \stackrel{\text{def}}{=} \sum_{i=1}^{N} y_i\lambda_i \qquad (1)$$

From the properties of the λ_i, we have $Q_1 = 1$, $Q_2 = x$ and $Q_3 = y$. Then the following polynomial in the λ_is is also a monomial $x^\sigma y^\tau$, $\sigma + \tau = p$:

$$Q_{\bar{m}}(\lambda) \stackrel{\text{def}}{=} \prod_{k=1}^{p} Q_{m_k}(\lambda) \qquad 2 \leq m_1 \leq \cdots \leq m_p \leq 3 \qquad (2)$$

where $\bar{m}$ is the multi-index $(m_1, m_2, \ldots m_p)$.

The basis set for the PFE of order p consists of one vertex function per node, λ_i, and $p-1$ edge functions per edge, ξ_e^q, $q = 2, \ldots, p$. Function ξ_e^q is a polynomial of degree q in the λ_i that vanishes on every edge except the one connecting nodes e and e+1. The basis of order p must be able to exactly represent $x^\sigma y^\tau$, $0 \leq \sigma + \tau \leq p$.

The basis is hierarchical. Suppose we know it for order $p-1$. To find the extra functions, ξ_e^p, needed for order p, we start by introducing another set, T_p, of known polynomials in the λ_i that can represent all $Q_{\bar{m}}(\lambda)$ in (2) and therefore all $x^\sigma y^\tau$ when $\sigma + \tau = p$. The set T_p consists of vertex functions; different edge functions, $\mu_i^q \stackrel{\text{def}}{=} \lambda_i\,\lambda_{i+1}\big(\lambda_i - \lambda_{i+1}\big)^{q-2}$; and interior functions, μ_{pj} (pth order polynomials in the λ_i, vanishing on the boundary). T_p is complete to order p in the λ_i and so we can express each $Q_{\bar{m}}(\lambda)$ in terms of it:

$$Q_{\bar{m}}(\lambda) = \sum_{i=1}^{N} G_{i\bar{m}}\,\lambda_i + \sum_{q=2}^{p}\sum_{i=1}^{N} G_{i\bar{m}}^q\,\mu_i^q + \sum_j G_{j\bar{m}}^I\,\mu_{pj} \qquad (3)$$

We have found an algorithm to find the quantities G in (3).

Next, we replace μ_i^q in (3) by ξ_i^q, which matches μ_i^q on the boundary but has additional, qth-order interior functions:

$$\xi_i^q \stackrel{\text{def}}{=} \mu_i^q + \sum_j C_{ij}^q\,\mu_{qj} \qquad (4)$$

Since the ξ_i^q are already known for $q < p$, so too are the coefficients C_{ij}^q. We need to find the C_{ij}^p. They are chosen so that, when μ_i^q replaces ξ_i^q in (3), the interior terms cancel, i.e., $Q_{\bar{m}}(\lambda)$ is a linear combination of just the basis functions λ_i and ξ_i^q. To do this, first we re-express each ξ_i^q in terms of pth order interior functions:

$$\xi_i^q = \mu_i^q + \sum_j \hat{C}_{ij}^q\,\mu_{pj} \qquad (5)$$

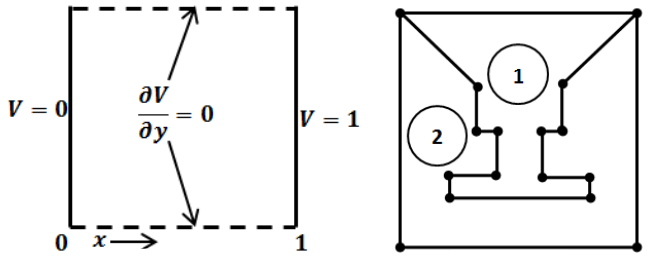
(with $\hat{C}_{ij}^p = C_{ij}^p$). After using (5) to replace μ_i^q by ξ_i^q in (3), the interior terms in (3) cancel provided:

$$\sum_{i=1}^{N} G_{i\bar{m}}^p\,C_{ij}^p = G_{j\bar{m}}^I - \sum_{q=2}^{p-1}\sum_{i=1}^{N} G_{i\bar{m}}^q\,\hat{C}_{ij}^q \qquad (6)$$

This has to hold for each of the $p+1$ monomials (2) and so for each interior function j, (6) is a set of $p+1$ coupled equations for the $N \geq p+1$ unknowns C_{ij}^p, $i = 1, \cdots, N$. When $N > p+1$, we have a systematic procedure for setting $N - (p+1)$ of the unknowns to zero and solving for the rest.

III. Results

The Helmholtz equation is solved for scalar V on a unit square domain, Ω, with a wave propagating in the x direction, Fig. 1 (a). The wave is time-harmonic, with wavenumber $k = 1$ rad/m.



(a) (b)

Fig. 1. (a) A square domain with wave propagation in the x direction. (b) A building block is formed of two concave PFEs, with $N = 14$ and $N = 16$.

The domain is subdivided into $2^n \times 2^n$, $n \in [0, 5]$, square building blocks. The building block is formed of two concave PFEs, numbered 1 and 2, with 14 and 16 nodes respectively as shown in Fig. 1(b). From the solution we evaluate the functional:

$$F = \frac{1}{2}\int_{\Omega} [(\nabla V)^2 - k^2 V^2]\, d\Omega \qquad (7)$$

Since we know the analytical solution to this problem, the error in the functional can be calculated.

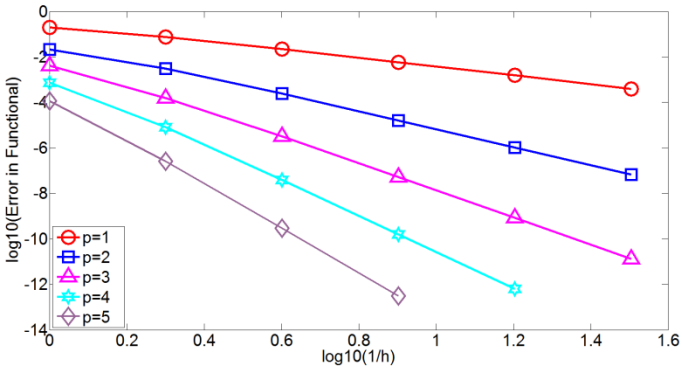


Fig. 2. Error in functional versus h^{-1} for building block shown in Fig 1(b). p is the element order.

TABLE I
SLOPE OF THE RIGHTMOST SECTION OF CURVE IN FIG. 2

p	1	2	3	4	5
Slope	−1.95	−3.98	−5.98	−7.94	−9.83

Fig. 2 is a logarithmic plot of the error in the functional for 1st to 5th order PFEs as a function of h^{-1}, where $h = 2^{-n}$ is the size of the building block. Table I shows the slope of the rightmost section of each curve. It confirms that for the p^{th} order element the error in the functional is $O(h^{2p})$, as expected.

Next, we solve Poisson's equation in a C-core problem (Fig. 3) to find the magnetostatic vector potential. The air region is modeled with three PFEs: 1, 2 and 6. The relative permeability of the C-core region (modeled with PFE 3) is 100. The C-core is excited by a coil, represented by two PFEs, 4 and 5, with current density J_z equal to $-0.5\ \text{A/m}^2$ and $0.5\ \text{A/m}^2$ respectively.

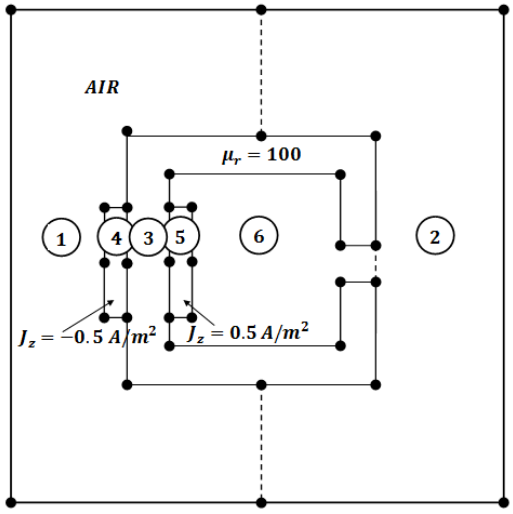


Fig. 3. C-core problem showing the six polygonal finite elements.

Table II gives the magnetic energy, the number of degrees of freedom (DOF) used to compute it and the maximum error in the potential, A, over a regular grid of 400 points, using 1st to 5th order PFEs. The reference value was obtained using a commercial FE system [5], with 25,848 triangular elements of order 4. From Table II, we see that as the PFE precision increases, the energy converges to the reference. Note that in the cases "Linear & quadratic", "Quadratic & cubic", etc., a mixture of element orders is used: elements 1, 4 and 5 are order p and elements 2, 3 and 6 are order $p + 1$.

TABLE II
MAGNETIC ENERGY AND ERROR IN POTENTIAL

Precision of PFEs	Energy ($\mu J/m$)	Max. error in A (nWb/m)	DOFs
All linear	6.933	14.53	32
Linear & quadratic	7.609	9.09	62
All quadratic	7.928	6.29	69
Quadratic & cubic	8.409	3.58	99
All cubic	8.464	2.78	106
Cubic & quartic	8.485	2.46	136
All quartic	8.503	2.32	143
Quartic & quintic	8.521	2.24	173
All quintic	8.527	2.21	180
Reference	8.542		

References

[1] T. Euler, R. Schuhmann, T. Weiland, "Polygonal finite elements," *IEEE Trans. on Magnetics*, vol.42, no.4, pp.675-678, April 2006.

[2] N. Sukumar, E. A. Malsch, "Recent advances in the construction of polygonal finite element interpolants," *Archives of Computational Methods in Engineering*, vol. 13, no. 1, pp.129-163, 2006.

[3] A. Rand, A. Gillette, C. Bajaj, "Quadratic serendipity finite elements on polygons using generalized barycentric coordinates," *Math. Comput.*, vol. 83, pp.2691-2716, 2014.

[4] T. Mukherjee, J. P. Webb, "Hierarchical bases for polygonal finite elements", *Sixteenth Bienniel IEEE Conference on Electromagnetic Field Computation*, Annecy, France, May 25-28, 2014; accepted July 30, 2014 for publication in *IEEE Trans. Magnetics*.

[5] MagNet 7.4.1, 2012, Infolytica Corporation, Montreal, Canada, http://www.infolytica.com.